## Electromagnetic Field Tensor

Electromagnetic field tensor is expressed in matrix form in 4x4 square matrix. The matrix elements are components of electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$. Firstly we will define electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$ in terms of magnetic vector potential $\vec{A}$ and scalar potential $\phi$ then after we will develope idea of this very tensor known as electromagnetic field tensor.

Electric field vector $\vec{E}$ and magnetic field vector $\vec{B}$ can be defined in terms of magnetic vector potential $\vec{A}$ and scalar potential $\phi$ as

$$\vec{E} = -\,\mathrm{grad}\,\phi - \frac{\partial \vec{A}}{\partial t} \qquad (1)$$

and $$\vec{B} = \mathrm{curl}\,\vec{A} \qquad (2)$$

Where $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} = E_1\hat{i} + E_2\hat{j} + E_3\hat{k}$

and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$.

From equⁿ (1),

$$E_1\hat{i} + E_2\hat{j} + E_3\hat{k} = -\frac{\partial\phi}{\partial x_1}\hat{i} - \frac{\partial\phi}{\partial x_2}\hat{j} - \frac{\partial\phi}{\partial x_3}\hat{k} - \frac{\partial A_1}{\partial t}\hat{i} - \frac{\partial A_2}{\partial t}\hat{j} - \frac{\partial A_3}{\partial t}\hat{k}$$

$$\Rightarrow E_1\hat{i} + E_2\hat{j} + E_3\hat{k} = -\hat{i}\left(\frac{\partial\phi}{\partial x_1} + \frac{\partial A_1}{\partial t}\right) - \hat{j}\left(\frac{\partial\phi}{\partial x_2} + \frac{\partial A_2}{\partial t}\right) - \hat{k}\left(\frac{\partial\phi}{\partial x_3} - \frac{\partial A_3}{\partial t}\right) \qquad (3)$$

From equⁿ (3), First component $E_1$ of electric field vector $\vec{E}$ is

$$E_1 = -\frac{\partial\phi}{\partial x_1} - \frac{\partial A_1}{\partial t} \qquad (4)$$

Now we are going to express RHS of equⁿ (4) in terms of four vector potential $A_\mu$. For this purpose, multiplying bothsides of equⁿ (4) by $\frac{i}{c}$.

$$\frac{i}{c}E_1 = -\frac{i}{c}\frac{\partial\phi}{\partial x_1} - \frac{i}{c}\frac{\partial A_1}{\partial t}$$

$$= -\frac{\partial\left(\frac{i}{c}\phi\right)}{\partial x_1} - i.i.\frac{\partial A_1}{\partial(ict)}$$

$$= -\frac{\partial A_4}{\partial x_1} + \frac{\partial A_1}{\partial x_4}$$

$\because \frac{i\phi}{c} = A_4 =$ 4rth component of $A_\mu$

$ict = x_4 =$ 4rth " " $x_\mu$

$i.i = i^2 = -1$

$$\Rightarrow \frac{i}{c} E_1 = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} \quad \text{———} \quad 5(a)$$

Similarly we can write with the help of eqn (3) and 5(a).

$$\frac{i}{c} E_2 = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} \quad \text{———} \quad 5(b)$$

$$\frac{i}{c} E_3 = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3} \quad \text{———} \quad 5(c)$$

From eqn (2). $\vec{B} = \text{curl}\,\vec{A} = \vec{\nabla} \times \vec{A}$

$$\Rightarrow B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k} = \left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right) \times (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$\Rightarrow B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$\Rightarrow B_1 \hat{i} + B_2 \hat{j} + B_3 \hat{k} = \hat{i}\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x_1} - \frac{\partial A_1}{\partial x_3}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \quad \text{———} \quad (6)$$

From eqn (6), components of magnetic field vector $\vec{B}$ will be

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} \quad \text{———} \quad 7(a)$$

$$B_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} \quad \text{———} \quad 7(b)$$

$$B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \quad \text{———} \quad 7(c)$$

We can write eqns (5) and (7) jointly in the form of single equation as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \quad \text{———} \quad (8)$$

with

$\frac{i}{c} E_1 = F_{41}$ | $F_{12} = B_3$

$\frac{i}{c} E_2 = F_{42}$ | $F_{23} = B_1$

$\frac{i}{c} E_3 = F_{43}$ | $F_{31} = B_2$

Now $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -\left(\frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu}\right) = -F_{\nu\mu}$

Thus $F_{\mu\nu} = -F_{\nu\mu} \Rightarrow$

$$F_{14} = -F_{41} = -\frac{i}{c}E_1 \qquad F_{21} = -F_{12} = -B_3$$
$$F_{24} = -F_{42} = -\frac{i}{c}E_2 \qquad F_{32} = -F_{23} = -B_1$$
$$F_{34} = -F_{43} = -\frac{i}{c}E_3 \qquad F_{13} = -F_{31} = -B_2$$

Again $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$

If we put $\mu = \nu$ then

$$F_{\mu\mu} = \frac{\partial A_\mu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\mu} = 0 \Rightarrow F_{\mu\mu} = 0 \Rightarrow F_{11} = F_{22} = F_{33} = F_{44} = 0$$

In matrix form, we may write $F_{\mu\nu}$ as

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

by putting different value of $F_{\mu\nu}$ from above, we get

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -\frac{i}{c}E_1 \\ -B_3 & 0 & B_1 & -\frac{i}{c}E_2 \\ B_2 & -B_1 & 0 & -\frac{i}{c}E_3 \\ \frac{i}{c}E_1 & \frac{i}{c}E_2 & \frac{i}{c}E_3 & 0 \end{bmatrix} \quad \text{——— (9)}$$

This quantity $F_{\mu\nu}$ has 16 components in 4D space and this quantity neither a scalar nor a vector but it is tensor of rank 2. This tensor $F_{\mu\nu}$ is known as electromagnetic field tensor.

Since $F_{\mu\nu} = -F_{\nu\mu}$ so $F_{\mu\nu}$ will be an antisymmetric tensor of rank 2. Since $F_{\mu\mu} = 0$ so $F_{\mu\nu}$ is known as skew symmetric tensor.